THE CHEN-RUAN COHOMOLOGY OF ALMOST CONTACT ORBIFOLDS

FAN DING, YUNFENG JIANG AND JIANZHONG PAN

Abstract. Comparing to the Chen-Ruan cohomology theory for the almost complex orbifolds, we study the orbifold cohomology theory for almost contact orbifolds. We define the Chen-Ruan cohomology group of any almost contact orbifold. Using the methods for almost complex orbifolds (see [2]), we define the obstruction bundle for any 3-multisector of the almost contact orbifolds and the Chen-Ruan cup product for the Chen-Ruan cohomology. We also prove that under this cup product the direct sum of all dimensional orbifold cohomology groups constitutes a cohomological ring. Finally we calculate two examples.

1. Introduction

Motivated by the orbifold string theory on the global quotient orbifolds studied by Dixon, Harvey, Vafa and Witten (see [4], [5]), and also Zaslow [14], Chen and Ruan [2] developed a very interesting cohomology theory for almost complex orbifolds which is now called Chen-Ruan cohomology. The remarkable aspect of this theory, besides the generalization of non global quotient orbifolds studied by physicists, is the existence of a ring structure obtained from Chen-Ruan's orbifold quantum cohomology construction (see [3]) by restricting to the class called ghost maps, the same as the ordinary cup product may be obtained by quantum cup product. The appearance of this interesting theory attracted many authors to calculate the Chen-Ruan cohomological rings of concrete examples, see [6],[12],[10],[7]. But we would like to point out that the definition of the Chen-Ruan cohomology is for almost complex orbifolds, in other words, the real dimensions of the orbiolds are even. In this paper we study the similar Chen-Ruan cohomology theory for almost contact orbifolds which have odd real dimensions. The motivation for us to study this theory for almost contact orbifolds is that the almost contact orbifolds are very related to almost complex orbifolds.

An almost contact orbifold X of dimension $2n + 1$ is an orbifold X together with an almost contact structure. And the actions of local groups of the orbifold preserve the almost contact structure. The most interesting feature of the orbifold cohomology for the almost contact orbifold is the cohomology of so-called twisted sectors defined by [8],[2]. For a twisted sector $X_{(g)}$, the property of almost contact structure implies that the action of g on the tangent space belongs to $U(n) \times 1$. Similar to the definition of the degree shifting number in [2], we define the degree shifting number $\iota_{(g)}$ of $X_{(g)}$. We have the following definition:

Definition 1.0.1. Let X be an almost contact orbifold of dimension $2n + 1$, we define the Chen-Ruan orbifold cohomology group of X by

$$H^d_{orb}(X; \mathbb{Q}) := \bigoplus_{(g) \in T_1} H^{d - 2\iota_{(g)}}(X_{(g)}, \mathbb{Q})$$

where T_1 represents all the equivalent classes of (g), which show the number of twisted sectors. For the almost contact orbifold X, $X \times \mathbb{R}$ is an almost complex orbifold. We call it the corresponding almost complex orbiofld of X. Let $H^d_{CR}(X \times \mathbb{R}; \mathbb{Q})$ represents the Chen-Ruan cohomology group defined in [2], we have the following result:

Theorem 1.0.2. *Let X be an almost contact orbifold of dimension $2n+1$, and $X \times \mathbb{R}$ be its corresponding almost complex orbifold, then as vector spaces, $H^d_{orb}(X; \mathbb{Q}) \cong H^d_{CR}(X \times \mathbb{R}; \mathbb{Q})$.*

To define the orbifold cup product on $H^*_{orb}(X; \mathbb{Q}) = \bigoplus_d H^d_{orb}(X; \mathbb{Q})$, we still adopt the same method of Chen and Ruan [2] using the degree zero Gromov-Witten invariant of the orbifold stable maps. Let $\eta_j \in H^{d_j}(X_{(g_j)}; \mathbb{Q})$ for $j = 1, 2$, $\eta_3 \in H^{d_3}_c(X_{(g_3)}; \mathbb{Q})$. we define $< \eta_1 \cup_{orb} \eta_2, \eta_3 >_{orb} := < \eta_1, \eta_2, \eta_3 >_{orb}$, where

$$< \eta_1, \eta_2, \eta_3 >_{orb} := \sum_{\mathbf{g} \in T_3^0} \int_{X_{(\mathbf{g})}}^{orb} e_1^* \eta_1 \wedge e_2^* \eta_2 \wedge e_3^* \eta_3 \wedge e_A(E_{(\mathbf{g})})$$

$X_{(\mathbf{g})} = X_{(g_1, g_2, g_3)}$ is a 3-multisector with $g_1 g_2 g_3 = 1$. The maps $e_j : X_{(\mathbf{g})} \longrightarrow X_{(g_j)}$ defined by $(p, (\mathbf{g})_p) \longmapsto (p, (g_j)_p)$ are the evaluation maps. $e_A(E_{(\mathbf{g})})$ is the Euler form of the obstruction bundle $E_{(\mathbf{g})}$ calculated from the connection A. The obstruction bundle $E_{(\mathbf{g})}$ over $X_{(\mathbf{g})}$ is defined from the obstruction bundle of the corresponding 3-multisector of its corresponding almost complex orbiofld $X \times \mathbb{R}$. Under this orbifold cup product, we have:

Theorem 1.0.3. *Let X be an almost contact orbifold of dimension $2n + 1$, then the orbifold cohomology $H^*_{orb}(X; \mathbb{Q})$ with the orbifold cup product defined above is a cohomological ring.*

Corollary 1.0.4. *Let X be an almost contact orbifold of dimension $2n + 1$, and $X \times \mathbb{R}$ be its corresponding almost complex orbifold, then as cohomological rings, $H^*_{orb}(X; \mathbb{Q}) \cong H^*_{CR}(X \times \mathbb{R}; \mathbb{Q})$.*

The paper is organized as follows. Section 2 is a review of some basic facts concerning orbifold, orbifold vector bundle and almost contact orbifold. In section 3 we define the orbifold cohomological ring of any almost contact orbifold. we introduce twisted sectors, degree shifting numbers and orbifold cup product for the almost contact orbifolds. Using the Chen-Ruan cup product of the corresponding almost complex orbifolds of the almost contact orbifolds, we prove that the orbifold cup product makes the orbifold cohomology of an almost contact orbifold into a cohomological ring. Finally in section 4 we give two examples.

2. Preliminaries

2.1. **Orbifold and orbifold vector bundle.** For the concept of orbifold, we know that the definition of Satake [11] is for reduced orbifolds. Here we introduce the general orbifolds which are not necessarily reduced. We give the definition from Chen and Ruan [2].

Definition 2.1.1. An orbifold structure on a Hausdorff, separate topological space X is given by an open cover $\mathcal{U}$ of X satisfying the following conditions.

(1) Each element U in $\mathcal{U}$ is uniformized, say by (V, G, π). Namely, V is a smooth manifold and G is a finite group acting smoothly on V such that $U = V/G$ with π as the quotient map. Let $Ker(G)$ be the subgroup of G acting trivially on V.

(2) For $U' \subset U$, there is a collection of injections $(V', G', \pi') \longrightarrow (V, G, \pi)$. Namely, the inclusion $i : U' \subset U$ can be lifted to maps $\widetilde{i} : V' \longrightarrow V$ and an injective homomorphism $i_* : G' \longrightarrow G$ such that i_* is an isomorphism from $Ker(G')$ to $Ker(G)$ and $\widetilde{i}$ is i_*-equivariant.

(3) For any point $x \in U_1 \cap U_2$, $U_1, U_2 \in \mathcal{U}$, there is a $U_3 \in \mathcal{U}$ such that $x \in U_3 \subset U_1 \cap U_2$.

For any point $x \in X$, suppose that (V, G, π) is a uniformizing neighborhood and $\overline{x} \in \pi^{-1}(x)$. Let G_x be the stabilizer of G at $\overline{x}$. Up to conjugation, it is independent of the choice of $\overline{x}$ and is called the *local group* of x. Then there exists a sufficiently small neighborhood V_x of $\overline{x}$ such that (V_x, G_x, π_x) uniformizes a small neighborhood of x, where π_x is the restriction $\pi \mid V_x$. (V_x, G_x, π_x) is called *a local chart* at x. The orbifold structure is called *reduced* if the action of G_x is effective for every x.

Let $pr : E \longrightarrow X$ be a rank k complex *orbifold bundle* over an orbifold X([2]). Then a uniformizing system for $E \mid U = pr^{-1}(U)$ over a uniformized subset U of X consists of the following data:

(1) A uniformizing system (V, G, π) of U.

(2) A uniformizing system $(V \times \mathbb{C}^k, G, \widetilde{\pi})$ for $E \mid U$. The action of G on $V \times \mathbb{C}^k$ is an extension of the action of G on V given by $g \cdot (x, v) = (g \cdot x, \rho(x, g)v)$ where $\rho : V \times G \longrightarrow Aut(\mathbb{C}^k)$ is a smooth map satisfying:

$$\rho(g \cdot x, h) \circ \rho(x, g) = \rho(x, hg), g, h \in G, x \in V.$$

(3)The natural projection map $\widetilde{pr} : V \times \mathbb{C}^k \longrightarrow V$ satisfies $\pi \circ \widetilde{pr} = pr \circ \widetilde{\pi}$.

By an orbifold connection $\triangle$ on E we mean an equivariant connection that satisfies $\triangle = g^{-1} \triangle g$ for every uniformizing system of E. Such a connection can be always obtained by averaging an equivariant partition of unity.

A C^∞ differential form on X is a G-invariant differential form on V for each uniformizing system (V, G, π). Then orbifold integration is defined as follows. Suppose $U = V/G$ is connected, for any compactly supported differential n-form ω on U, which is, by definition, a G-invariant n-form $\widetilde{\omega}$ on V,

$$\int_U^{orb} \omega := \frac{1}{|G|} \int_V \widetilde{\omega} \tag{2.1}$$

Where $|G|$ is the order of G. The orbifold integration over X is defined by using a C^∞ partition of unity. The orbifold integration coincides with the usual measure theoretic integration iff the orbifold structure is reduced.

2.2. **The Almost Contact Orbifolds.** In this section we introduce the basic concept of almost contact orbifolds. First we give the definition of an almost contact orbifold.

Definition 2.2.1. An almost contact structure on a orbifold X of dimension $2n+1$ is a co-dimension one distribution ξ on X which is locally given by the kernel of a 1-form η such that ξ has an almost complex structure J, and for every uniformizing system (V, G, π), the action of G on V preserves this almost complex structure J.

Definition 2.2.2. A contact structure on a orbifold X of dimension $2n + 1$ is a co-dimension one distribution ξ on X which is given by the kernel of a global 1-form η with $\eta \wedge d\eta \neq 0$. η is said to represent ξ, and is called a contact form.

For the next part of the paper, an almost contact orbifold is always an orbifold such that the distribution ξ has an almost complex structure. So the tangent bundle of X has the structure group $U(n) \times 1$. Given an almost contact orbifold X. For an open subset U of X, let (V, G, π) be its local uniformizing neighborhood. If $g \in G$ is an element of G, then the action of g on the open set V belongs to $U(n) \times 1$.

Remark 2.2.3. For a contact orbifold X with contact form η, the symplectization of X is a symplectic orbifold. From [13], for the co-oriented orbifold X, its symplectization can be identified with $(X \times \mathbb{R}, d(e^t\eta))$, where t is the $\mathbb{R}$ coordinate.

3. The Chen-Ruan Cohomology Rings of Almost Contact Orbifolds

3.1. **Twisted Sectors and the Chen-Ruan Cohomology Group.** First we introduce twisted sectors. Let X be an orbifold. Consider the set of pairs:

$$\widetilde{X}_k = \{(p, (\mathbf{g})_{G_p}) | p \in X, \mathbf{g} = (g_1, \cdots, g_k), g_i \in G_p\}$$

where $(\mathbf{g})_{G_p}$ is the conjugacy class of k-tuple $\mathbf{g} = (g_1, \cdots, g_k)$ in G_p. We use G^k to denote the set of k-tuples. If there is no confusion, we will omit the subscript G_p to simplify the notation. Suppose that X has an orbifold structure $\mathcal{U}$ with uniformizing systems $\{(\widetilde{U}, G_U, \pi_U)\}$. From Chen and Ruan [2], also see [8], $\widetilde{X}_k$ is naturally an orbifold, with the generalized orbifold structure at $(p, (\mathbf{g})_{G_p})$ given by $(V_p^{\mathbf{g}}, C(\mathbf{g}), \pi : V_p^{\mathbf{g}} \longrightarrow V_p^{\mathbf{g}}/C(\mathbf{g}))$, where $V_p^{\mathbf{g}} = V_p^{g_1} \cap \cdots V_p^{g_k}$, $C(\mathbf{g}) = C(g_1) \cap \cdots C(g_k)$. Here $\mathbf{g} = (g_1, \cdots, g_k)$, V_p^g stands for the fixed point set of g in V_p. When X is a almost contact orbifold of dimension $2n + 1$, $\widetilde{X}_k$ inherits a almost contact structure from X, and when X is closed, $\widetilde{X}_k$ is finite disjoint union of closed orbifolds.

Now we describe the the connected components of $\widetilde{X}_k$, Recall that every point p has a local chart (V_p, G_p, π_p) which gives a local uniformized neighborhood $U_p = \pi_p(V_p)$. If $q \in U_p$, up to conjugation there is a unique injective homomorphism $i_* : G_q \longrightarrow G_p$. For $\mathbf{g} \in (G_q)^k$, the conjugation class $i_*(\mathbf{g})_q$ is well defined. We define an equivalence relation $i_*(\mathbf{g})_q \cong (\mathbf{g})_q$. Let T_k denote the set of equivalence classes. To abuse the notation, we use $(\mathbf{g})$ to denote the equivalence class which $(\mathbf{g})_q$ belongs to. We will usually denote an element of T_1 by (g). It is clear that $\widetilde{X}_k$ can be decomposed as a disjoint union of connected components:

$$\widetilde{X}_k = \bigsqcup_{(\mathbf{g}) \in T_k} X_{(\mathbf{g})}$$

Where $X_{(\mathbf{g})} = \{(p, (\mathbf{g}')_p) | \mathbf{g}' \in (G_p)^k, (\mathbf{g}')_p \in (\mathbf{g})\}$. Note that for $\mathbf{g} = (1, \cdots, 1)$, we have $X_{(\mathbf{g})} = X$. A component $X_{(\mathbf{g})}$ is called a $k - multisector$, if $\mathbf{g}$ is not the

identity. A component of $X_{(g)}$ is simply called a *twisted sector*. If X has a almost contact structure or an almost complex, then $X_{(\mathbf{g})}$ has the analogous structure induced from X. We define

$$T_3^0 = \{(\mathbf{g}) = (g_1, g_2, g_3) \in T_3 | g_1 g_2 g_3 = 1\}.$$

Note that there is an one to one correspondence between T_2 and T_3^0 given by $(g_1, g_2) \longmapsto (g_1, g_2, (g_1 g_2)^{-1})$.

Now we define the Chen-Ruan cohomology group. Let X be an almost contact orbifold of dimension $2n + 1$. Then for a point p with nontrivial group G_p, from section 2.2, we know that the almost contact structure on X gives rise to an effective representation $\rho_p : G_p \longrightarrow U(n, \mathbb{C}) \times 1$. For any $g \in G_p$, we write $\rho_p(g)$, up to conjugation, as a diagonal matrix

$$diag\left(e^{2\pi i \frac{m_{1,g}}{m_g}}, \ldots, e^{2\pi i \frac{m_{n,g}}{m_g}}\right) \times 1.$$

where m_g is the order of g in G_p, and $0 \leq m_{i,g} < m_g$. Define a function $\iota : \widetilde{X_1} \longrightarrow \mathbb{Q}$ by

$$\iota(p, (g)_p) = \sum_{i=1}^{n} \frac{m_{i,g}}{m_g}.$$

We can see that the function $\iota : \widetilde{X_1} \longrightarrow \mathbb{Q}$ is locally constant and $\iota = 0$ if $g = 1$. Denote its value on $X_{(g)}$ by ι_g. We call ι_g the degree shifting number of $X_{(g)}$. It has the following properties:

(1) $\iota_{(g)}$ is an integer iff $\rho_p(g) \in SL(n, \mathbb{C}) \times 1$;
(2) $2(\iota_{(g)} + \iota_{(g^{-1})}) = 2rank(\rho_p(g) - Id \times 1) = 2n - dim_{\mathbb{R}} X_{(g)}$.

Definition 3.1.1. Let X be an almost contact orbifold of dimension $2n + 1$, we define the orbifold cohomology group of X by

$$H_{orb}^d(X; \mathbb{Q}) := \bigoplus_{(g) \in T_1} H^{d - 2\iota_{(g)}}(X_{(g)}, \mathbb{Q})$$

Let $X \times \mathbb{R}$ be the corresponding almost complex orbifold of X, then it is easy to see that the orbifold structure of $X \times \mathbb{R}$ can be easily described. Let $Y = X \times \mathbb{R}$, then a k-multisector $Y_{(\mathbf{g})} = X_{(\mathbf{g})} \times \mathbb{R}$ for $\mathbf{g} = (g_1, \cdots, g_k)$. We have the following theorem:

Theorem 3.1.2. *Let X be an almost contact orbifold of dimension $2n+1$, and $X \times \mathbb{R}$ be its corresponding almost complex orbifold , then as vector spaces, $H_{orb}^d(X; \mathbb{Q}) \cong H_{CR}^d(X \times \mathbb{R}; \mathbb{Q})$.*

Proof. We see that there exists an one to one correspondence between the twisted sectors of the orbifolds X and $X \times \mathbb{R}$, and this correspondence is given by $X_{(\mathbf{g})} \longrightarrow Y_{(\mathbf{g})} = X_{(\mathbf{g})} \times \mathbb{R}$. The degree shifting numbers of the two twisted sectors are equal, and any dimensional cohomology groups of X and $X \times \mathbb{R}$ are isomorphic. From the definition of the Chen-Ruan cohomology group of $X \times \mathbb{R}$ in [2] and definition 3.1.1, the theorem is proved. $\square$

3.2. **The Obstruction Bundle.** Let X be an almost contact orbifold of dimension $2n + 1$, we consider its corresponding almost complex orbifold $Y = X \times \mathbb{R}$. For a 3-mutisector $X_{(\mathbf{g})} = X_{(g_1, g_2, g_3)}$ with $(g_1, g_2, g_3) \in T_3^0$, we know that $Y_{(\mathbf{g})} = X_{(\mathbf{g})} \times \mathbb{R}$. Let $((p, x), (\mathbf{g})_{(p,x)})$ be a generic point in $X_{(\mathbf{g})} \times \mathbb{R}$. Let $K(\mathbf{g})$ be the subgroup

of G_p generated by g_1 and g_2. Consider an orbifold Riemann sphere with three orbifold points $(S^2, (p_1, p_2, p_3), (k_1, k_2, k_3))$. When there is no confusion, we will simply denote it by S^2. The orbifold fundamental group is:

$$\pi_1^{orb}(S^2) = \{\lambda_1, \lambda_2, \lambda_3 | \lambda_i^{k_i} = 1, \lambda_1 \lambda_2 \lambda_3 = 1\}$$

Where λ_i is represented by a loop around the marked p_i. There is a surjective homomorphism

$$\rho : \pi_1^{orb}(S^2) \longrightarrow K(\mathbf{g})$$

specified by mapping $\lambda_i \longmapsto g_i$. $Ker(\rho)$ is a finite-index subgroup of $\pi_1^{orb}(S^2)$. Let $\widetilde{\Sigma}$ be the orbifold universal cover of S^2. Let $\Sigma = \widetilde{\Sigma}/Ker(\rho)$. Then Σ is smooth, compact and $\Sigma/K(\mathbf{g}) = S^2$. The genus of Σ can be computed using Riemann Hurwitz formula for Euler characteristics of a branched covering, and turns out to be

$$g(\Sigma) = \frac{1}{2}(2 + |K(\mathbf{g})| - \Sigma_{i=1}^3 \frac{|K(\mathbf{g})|}{k_i}) \tag{3.1}$$

$K(\mathbf{g})$ acts holomorphically on Σ and hence $K(\mathbf{g})$ acts on $H^{0,1}(\Sigma)$. The "obstruction bundle" $\widetilde{E}_{(\mathbf{g})}$ over $X_{(\mathbf{g})} \times \mathbb{R}$ is constructed as follows. On the local chart $(V_p^{\mathbf{g}} \times \mathbb{R}, C(\mathbf{g}), \pi)$ of $X_{(\mathbf{g})} \times \mathbb{R}$, $\widetilde{E}_{(\mathbf{g})}$ is given by $(T(V_p \times \mathbb{R}) \otimes H^{0,1}(\Sigma))^{K(\mathbf{g})} \times (V_p^{\mathbf{g}} \times \mathbb{R}) \longrightarrow V_p^{\mathbf{g}} \times \mathbb{R}$, where $(T(V_p \times \mathbb{R}) \otimes H^{0,1}(\Sigma))^{K(\mathbf{g})}$ is the $K(\mathbf{g})$-invariant subspace. We define an action of $C(\mathbf{g})$ on $T(V_p \times \mathbb{R}) \otimes H^{0,1}(\Sigma)$, which is the usual one on $T(V_p \times \mathbb{R})$ and trivial on $H^{0,1}(\Sigma)$. The the action of $C(\mathbf{g})$ and $K(\mathbf{g})$ commute and $(T(V_p \times \mathbb{R}) \otimes H^{0,1}(\Sigma))^{K(\mathbf{g})}$ is invariant under $C(\mathbf{g})$. Thus we have obtained an action of $C(\mathbf{g})$ on $(T(V_p \times \mathbb{R}) \otimes H^{0,1}(\Sigma))^{K(\mathbf{g})} \times (V_p^{\mathbf{g}} \times \mathbb{R}) \longrightarrow V_p^{\mathbf{g}} \times \mathbb{R}$, extending the usual one on $V_p^{\mathbf{g}} \times \mathbb{R}$. These trivializations fit together to define the bundle $\widetilde{E}_{(\mathbf{g})}$ over $X_{(\mathbf{g})} \times \mathbb{R}$. If we set $e \times 1 : X_{(\mathbf{g})} \times \mathbb{R} \longrightarrow X \times \mathbb{R}$ to be the map given by $((p, x), (\mathbf{g})_{(p,x)}) \longmapsto (p, x)$, one may think of $\widetilde{E}_{(\mathbf{g})}$ as $((e \times 1)^* T(X \times \mathbb{R}) \otimes H^{0,1}(\Sigma))^{K(\mathbf{g})}$. The rank of $\widetilde{E}_{(\mathbf{g})}$ is given by the formula [2]:

$$rank_{\mathbb{R}}(\widetilde{E}_{(\mathbf{g})}) = dim_{\mathbb{R}}(X_{(\mathbf{g})} \times \mathbb{R}) - dim_{\mathbb{R}}(X \times \mathbb{R}) + 2\Sigma_{j=1}^3 \iota_{(g_j)} \tag{3.2}$$

Let $e : X_{(\mathbf{g})} \longrightarrow X$ to be the map given by $(p, (\mathbf{g})_p) \longmapsto p$, we have the following proposition:

Proposition 3.2.1. Let $\pi_{(\mathbf{g})} : X_{(\mathbf{g})} \times \mathbb{R} \longrightarrow X_{(\mathbf{g})}$ be the natural projection, then $((e \times 1)^* T(X \times \mathbb{R}) \otimes H^{0,1}(\Sigma))^{K(\mathbf{g})} \cong \pi_{(\mathbf{g})}^* ((e^* TX \otimes H^{0,1}(\Sigma))^{K(\mathbf{g})})$

Proof. We have

$$
\begin{aligned}
((e \times 1)^* T(X \times \mathbb{R}) \otimes H^{0,1}(\Sigma))^{K(\mathbf{g})} &= \pi_{(\mathbf{g})}^* ((e^* TX \otimes H^{0,1}(\Sigma)) \oplus (T\mathbb{R} \otimes H^{0,1}(\Sigma)))^{K(\mathbf{g})} \\
&= \pi_{(\mathbf{g})}^* ((e^* TX \otimes H^{0,1}(\Sigma))^{K(\mathbf{g})} \oplus (T\mathbb{R} \otimes H^{0,1}(\Sigma))^{K(\mathbf{g})}) \\
&= \pi_{(\mathbf{g})}^* (e^* TX \otimes H^{0,1}(\Sigma))^{K(\mathbf{g})}
\end{aligned}
$$

The third equality is right because $K(\mathbf{g})$ acts on $T\mathbb{R}$ trivially, and $(H^{0,1}(\Sigma))^{K(\mathbf{g})} = H^{0,1}(S^2) = 0$. So we complete the proof of the proposition. □

We define the obstruction bundle $E_{(\mathbf{g})}$ over $X_{(\mathbf{g})}$ as $\left(e^*TX \otimes H^{0,1}(\Sigma)\right)^{K(\mathbf{g})}$. It is easy to see that the rank of $E_{(\mathbf{g})}$ is equal to the rank of $\widetilde{E}_{(\mathbf{g})}$. The rank is:

$$rank_{\mathbb{R}}(E_{(\mathbf{g})}) = dim_{\mathbb{R}}(X_{(\mathbf{g})}) - dim_{\mathbb{R}}(X) + 2\Sigma_{j=1}^3 \iota_{(g_j)} \tag{3.3}$$

3.3. **The Chen-Ruan Cup Product.** For an almost contact orbifold X, there is a natural map $I : X_{(g)} \longrightarrow X_{(g^{-1})}$ defined by $(p, (g)_p) \longmapsto (p, (g^{-1})_p)$.

Definition 3.3.1. (Poincaré Duality) Let X be an almost contact orbifold of dimension $2n+1$. For any $0 \leq d \leq 2n$, the pairing

$$<, >_{orb}: H^d_{orb}(X) \times H^{2n-d}_{orb,c}(X) \longrightarrow \mathbb{Q}$$

is defined by taking the direct sum of

$$<, >_{orb}^{(g)}: H^{d-2\iota_{(g)}}(X_{(g)};\mathbb{Q}) \times H_c^{2n-d-2\iota_{(g^{-1})}}(X_{(g^{-1})};\mathbb{Q}) \longrightarrow \mathbb{Q}$$

where

$$< \alpha, \beta >_{orb}^{(g)} = \int_{X_{(g)}}^{orb} \alpha \wedge I^*(\beta)$$

for $\alpha \in H^{d-2\iota_{(g)}}(X_{(g)};\mathbb{Q})$, and $\beta \in H^{2n-d-2\iota_{(g^{-1})}}(X_{(g^{-1})};\mathbb{Q})$.

Choose an orbifold connection A on $E_{(\mathbf{g})}$. Let $e_A(E_{(\mathbf{g})})$ be the Euler form computed from the connection A by Chen-Weil theory. Let $\eta_j \in H^{d_j}(X_{(g_j)};\mathbb{Q})$, for $j = 1, 2$, $\eta_3 \in H_c^{d_3}(X_{(g_3)};\mathbb{Q})$. Define maps $e_j : X_{(\mathbf{g})} \longrightarrow X_{(g_j)}$ by $(p, (\mathbf{g})_p) \longmapsto (p, (g_j)_p)$.

Definition 3.3.2. Similar to the definition in [2], we define the 3-point function to be

$$< \eta_1, \eta_2, \eta_3 >_{orb} := \sum_{\mathbf{g} \in T_3^0} \int_{X_{(\mathbf{g})}}^{orb} e_1^*\eta_1 \wedge e_2^*\eta_2 \wedge e_3^*\eta_3 \wedge e_A(E_{(\mathbf{g})}) \tag{3.4}$$

Note that the above integral does not depend on the choice of A. As in the definition 3.1.1, we extend the 3-point function to $H^*_{orb}(X)$ by linearity. We define the orbifold cup product by the relation

$$< \eta_1 \cup_{orb} \eta_2, \eta_3 >_{orb} := < \eta_1, \eta_2, \eta_3 >_{orb} \tag{3.5}$$

Again we extend $\cup_{orb}$ to $H^*_{orb}(X)$ via linearity. Note that if $(\mathbf{g}) = (1, 1, 1)$, then $\eta_1 \cup_{orb} \eta_2$ is just the ordinary cup product $\eta_1 \cup \eta_2$ in $H^*(X)$.

3.4. **Associativity.** In this section we prove the associativity of the Chen-Ruan cup product defined in the section 3.3. What we do here is to compare the CR-cup product of $H^*_{orb}(X)$ with the orbifold cup product of the chen-Ruan cohomology of its corresponding almost complex orbifold $X \times \mathbb{R}$ defined in [2]. We have the following theorem:

Theorem 3.4.1. *Let X be an almost contact orbifold of dimension $2n+1$, then the orbifold cohomology $H^*_{orb}(X;\mathbb{Q})$ with the Chen-Ruan cup product defined in section 3.3 is a cohomological ring.*

Proof. From section 3.1, we see that there is an one to one correspondence between the twisted sectors of X and its corresponding almost complex orbifold $Y = X \times \mathbb{R}$. By definition 3.1.1 and the definition of the orbifold cohomology of $X \times \mathbb{R}$ defined in [2], we have:

$$H^*_{orb}(X; \mathbb{Q}) := \bigoplus_{(g) \in T_1} H^{*-2\iota_{(g)}}(X_{(g)}, \mathbb{Q})$$

and

$$H^*_{orb}(X \times \mathbb{R}; \mathbb{Q}) := \bigoplus_{(g) \in T_1} H^{*-2\iota_{(g)}}(X_{(g)} \times \mathbb{R}, \mathbb{Q})$$

For the k-multisector $X_{(\mathbf{g})}$, we know that $Y_{(\mathbf{g})} = X_{(\mathbf{g})} \times \mathbb{R}$. Let $\pi_{(\mathbf{g})} : X_{(\mathbf{g})} \times \mathbb{R} \longrightarrow X_{(\mathbf{g})}$ be the natural projection. Then from [1], $(\pi_{(\mathbf{g})})_* : H^*_{cv}(X_{(\mathbf{g})} \times \mathbb{R}) \longrightarrow H^{*-1}(X_{(\mathbf{g})})$ is an isomorphism.

Let $X_{(\mathbf{g})}$ be a 3-multisector with $g_1 g_2 g_3 = 1$. For any $\eta_1 \in H^*(X_{(g_1)})$, $\eta_2 \in H^*(X_{(g_2)})$, $\eta_3 = (\pi_{(g_3)})_* \gamma \in H^*(X_{(g_3)})$, where $\gamma \in H^*_{cv}(X_{(\mathbf{g})} \times \mathbb{R})$. We have the following commutative diagram:

$$
\begin{array}{ccc}
X_{(\mathbf{g})} \times \mathbb{R} & \xrightarrow{\ \pi_{(\mathbf{g})}\ } & X_{(\mathbf{g})} \\
e_i \times 1 \downarrow & & e_i \downarrow \\
X_{(g_i)} \times \mathbb{R} & \xrightarrow{\ \pi_{(g_i)}\ } & X_{(g_i)}
\end{array}
$$

for $i = 1, 2, 3$. So we have $(e_i \times 1)^*(\pi_{(g_i)})^* \eta_i = (\pi_{(\mathbf{g})})^* e_i^* \eta_i$ for i=1,2 and $(\pi_{(\mathbf{g})})_*(e_3 \times 1)^* \gamma = e_3^*(\pi_{(g_3)})_* \gamma$. From the definition of the 3-point function for almost complex orbifolds in [2] and the orbifold version of the proposition 6.15 in [1], we have:

$$< (\pi_{(g_1)})^* \eta_1, (\pi_{(g_2)})^* \eta_2, \gamma >_{orb}$$

$$= \sum_{\mathbf{g} \in T_3^0} \int_{X_{(\mathbf{g})} \times \mathbb{R}}^{orb} (e_1 \times 1)^*(\pi_{(g_1)})^* \eta_1 \wedge (e_2 \times 1)^*(\pi_{(g_2)})^* \eta_2 \wedge (e_3 \times 1)^* \gamma \wedge e_{\widetilde{A}}(\widetilde{E}_{(\mathbf{g})})$$

$$= \sum_{\mathbf{g} \in T_3^0} \int_{X_{(\mathbf{g})} \times \mathbb{R}}^{orb} (\pi_{(\mathbf{g})})^* e_1^* \eta_1 \wedge (\pi_{(\mathbf{g})})^* e_2^* \eta_2 \wedge (e_3 \times 1)^* \gamma \wedge (\pi_{(\mathbf{g})})^* e_A(E_{(\mathbf{g})})$$

$$= (-1)^{deg\gamma \cdot deg e_A(E_\mathbf{g})} \sum_{\mathbf{g} \in T_3^0} \int_{X_{(\mathbf{g})} \times \mathbb{R}}^{orb} (\pi_{(\mathbf{g})})^*(e_1^* \eta_1 \wedge e_2^* \eta_2 \wedge e_A(E_{(\mathbf{g})})) \wedge (e_3 \times 1)^* \gamma$$

$$= (-1)^{deg\gamma \cdot deg e_A(E_\mathbf{g})} \sum_{\mathbf{g} \in T_3^0} \int_{X_{(\mathbf{g})}}^{orb} e_1^* \eta_1 \wedge e_2^* \eta_2 \wedge e_A(E_{(\mathbf{g})}) \wedge (\pi_{(\mathbf{g})})_*(e_3 \times 1)^* \gamma$$

$$= - \sum_{\mathbf{g} \in T_3^0} \int_{X_{(\mathbf{g})}}^{orb} e_1^* \eta_1 \wedge e_2^* \eta_2 \wedge e_3^*(\pi_{(g_3)})_* \gamma \wedge e_A(E_{(\mathbf{g})})$$

$$= - < \eta_1, \eta_2, (\pi_{(g_3)})_* \gamma >_{orb}$$

We see that the orbifold cup product of the Chen-Ruan cohomology $H^*_{orb}(X; \mathbb{Q})$ is identified with the CR-product of the orbifold cohomology for the corresponding almost complex orbifold $X \times \mathbb{R}$ of X module a sign. From [2], the CR-product for the almost complex orbifold $X \times \mathbb{R}$ satisfies the associativity, so the orbifold cup product of the Chen-Ruan cohomology $H^*_{orb}(X; \mathbb{Q})$ for the almost contact orbifold

X also satisfies the associativity. So the Chen-Ruan cohomology $H^*_{orb}(X; \mathbb{Q})$ for the almost contact orbifold X with this orbifold cup product forms a cohomological ring. This complete the proof of the theorem. $\qquad\square$

From the proof of the above theorem, we have the following corollary:

Corollary 3.4.2. *Let X be an almost contact orbifold of dimension $2n + 1$, and $X \times \mathbb{R}$ be its corresponding almost complex orbifold, then as cohomological rings, $H^*_{orb}(X; \mathbb{Q}) \cong H^*_{CR}(X \times \mathbb{R}; \mathbb{Q})$.*

4. EXAMPLES.

4.1. **Example 4.1.** The 3-sphere $X = S^3$ is an almost contact manifold with standard almost contact form. Let $\mathbb{Z}_3$ be a cyclic group of order 3. suppose that $\mathbb{Z}_3$ acts on S^3 in a plane and preserves the almost contact form, then the quotient $S^3/\mathbb{Z}_3$ is a almost contact orbifold of dimension 3. We see that this orbifold has one singular submanifold S^1, and the local orbifold group in this singular set is the cyclic group $\mathbb{Z}_3$. Let $g = e^{2\pi i \frac{1}{3}} \in \mathbb{Z}_3$, then $X_{(g)} = S^1$ and $X_{(g^2)} = S^1$, and the degree shifting numbers for the two twisted sectors are $\frac{1}{3}$ and $\frac{2}{3}$. So from the definition 3.1.1, we have:

$$H^d_{orb}(S^3/\mathbb{Z}_3; \mathbb{Q}) = H^d(S^3/\mathbb{Z}_3; \mathbb{Q}) \bigoplus H^{d-\frac{2}{3}}(S^1; \mathbb{Q}) \bigoplus H^{d-\frac{4}{3}}(S^1; \mathbb{Q})$$

For $\mathbf{g}_1 = (g, g, g) \in T^0_3$, we have $X_{(\mathbf{g})} = S^1$, and $X_{\mathbf{g}_2} = X_{(g^2, g^2, g^2)} = S^1$. From the formula (3.3), we see that the rank of the obstruction bundle $E_{(\mathbf{g}_1)}$ over $X_{(\mathbf{g}_1)}$ is 0, the rank of the obstruction bundle $E_{(\mathbf{g}_2)}$ over $X_{(\mathbf{g}_2)}$ is 2. So for $\eta_i \in H^*(X_{(g)}; \mathbb{Q})(i = 1, 2, 3)$, from section 3.3, we have

$$< \eta_1, \eta_2, \eta_3 >_{orb} := \int_{X_{(\mathbf{g}_1)}}^{orb} e_1^* \eta_1 \wedge e_2^* \eta_2 \wedge e_3^* \eta_3$$

The integration is the usual integration on the orbifold and we can calculate easily.
For $\eta_i \in H^*(X_{(g^2)}; \mathbb{Q})(i = 1, 2, 3)$, from section 3.3, we have

$$< \eta_1, \eta_2, \eta_3 >_{orb} := \int_{X_{(\mathbf{g}_2)}}^{orb} e_1^* \eta_1 \wedge e_2^* \eta_2 \wedge e_3^* \eta_3 \wedge e_A(E_{(\mathbf{g}_2)})$$

But the dimension of the Euler class $e_A(E_{(\mathbf{g}_2)})$ is 2, while $X_{(\mathbf{g}_2)}$ has dimension 1, so the above 3-point function is zero.

For the 3-multisector $X_{(g, g^2, 1)}$, it is easy to calculate that the obstruction bundle over it has dimension zero. The 3-point function is the usual integration on orbifold and we can easily calculate it. So we complete the analysis the Chen-Ruan cohomology ring of the almost contact orbifold $S^3/\mathbb{Z}_3$.

4.2. **Example 4.2.** We consider the weighted projective space $X = \mathbb{P}(1, 2, 2, 3, 3, 3)$. From the example of [7], we know that X is an orbifold. And the twisted sectors of this orbifold are $X_{(g_2)} = \mathbb{P}(2, 2)$, $X_{(g_1)} = \mathbb{P}(3, 3, 3)$ and $X_{(g_1^2)} = \mathbb{P}(3, 3, 3)$, the degree shifting numbers are $\iota_{(g_1)} = \frac{5}{3}$, $\iota_{(g_1^2)} = \frac{4}{3}$ and $\iota_{(g_2)} = 2$. Let S^1 be the 1-dimensional sphere. Then $Y = X \times S^1 = \mathbb{P}(1, 2, 2, 3, 3, 3) \times S^1$ is an almost contact orbifold. It is easy to see that the twisted sectors of Y are $Y_{(g_2 \times 1)} = \mathbb{P}(2, 2) \times S^1$,

$Y_{(g_1 \times 1)} = \mathbb{P}(3,3,3) \times S^1$ and $Y_{(g_1^2 \times 1)} = \mathbb{P}(3,3,3) \times S^1$ and the degree shifting numbers are $\iota_{(g_1 \times 1)} = \frac{5}{3}$, $\iota_{(g_1^2 \times 1)} = \frac{4}{3}$ and $\iota_{(g_2 \times 1)} = 2$. So the Chen-Ruan cohomology group of $Y = \mathbb{P}(1,2,2,3,3,3) \times S^1$ is

$$
\begin{aligned}
H_{orb}^d(\mathbb{P}(1,2,2,3,3,3) \times S^1; \mathbb{Q}) &= H^d(\mathbb{P}(1,2,2,3,3,3) \times S^1; \mathbb{Q}) \\
&\oplus H^{d-\frac{10}{3}}(\mathbb{P}(3,3,3) \times S^1; \mathbb{Q}) \\
&\oplus H^{d-\frac{8}{3}}(\mathbb{P}(3,3,3) \times S^1; \mathbb{Q}) \\
&\oplus H^{d-4}(\mathbb{P}(2,2) \times S^1; \mathbb{Q})
\end{aligned}
$$

For the Chen-Ruan ring structure, we see that the 3-multisectors are $Y_{(g_1 \times 1, g_1 \times 1, g_1 \times 1)}$ $= Y_{(g_1^2 \times 1, g_1^2 \times 1, g_1^2 \times 1)} = \mathbb{P}(3,3,3) \times S^1$, $Y_{(g_1 \times 1, g_1^2 \times 1, 1)} = \mathbb{P}(3,3,3) \times S^1$ and $Y_{(g_2 \times 1, g_2 \times 1, 1)}$ $= \mathbb{P}(2,2) \times S^1$. For the 3-multisectors $Y_{(g_1 \times 1, g_1^2 \times 1, 1)}$ and $Y_{(g_2 \times 1, g_2 \times 1, 1)}$, it is easily to see that the dimensions of the obstruction bundles over these 3-multisectors are zero, so the integrations of the 3-point function (3.4) are the usual integration on the orbifolds, we can determine them easily.

For the 3-multisector $Y_{(g_1 \times 1, g_1 \times 1, g_1 \times 1)} = \mathbb{P}(3,3,3) \times S^1$, let $\eta_i \in H^*(\mathbb{P}(3,3,3) \times S^1; \mathbb{Q})$ for $i = 1, 2, 3$. From (3.3), the dimension of the obstruction bundle $E_{(\mathbf{g} \times 1)}$ over $Y_{(g_1 \times 1, g_1 \times 1, g_1 \times 1)}$ is 4. The integration

$$
< \eta_1, \eta_2, \eta_3 >_{orb} := \int_{\mathbb{P}(3,3,3) \times S^1}^{orb} e_1^* \eta_1 \wedge e_2^* \eta_2 \wedge e_3^* \eta_3 \wedge e_A(E_{(\mathbf{g} \times 1)}) \qquad (4.1)
$$

is nonzero only if there is some $\eta_j \in H^1(\mathbb{P}(3,3,3) \times S^1; \mathbb{Q})$. We assume that $\eta_1 \in H^1(\mathbb{P}(3,3,3) \times S^1; \mathbb{Q}), \eta_2 \in H^0(\mathbb{P}(3,3,3) \times S^1; \mathbb{Q})$, $\eta_3 \in H^0(\mathbb{P}(3,3,3) \times S^1; \mathbb{Q})$. So the integration (4.1) is

$$
< \eta_1, \eta_2, \eta_3 >_{orb} := \int_{\mathbb{P}(3,3,3) \times S^1}^{orb} e_1^* \eta_1 \wedge e_A(E_{(\mathbf{g} \times 1)}) \qquad (4.2)
$$

Because $H^1(\mathbb{P}(3,3,3) \times S^1; \mathbb{Q}) = H^0(\mathbb{P}(3,3,3); \mathbb{Q}) \otimes H^1(S^1; \mathbb{Q})$, $e_1^* \eta_1$ represents the class of $H^1(S^1; \mathbb{Q})$. $e_1^*(\eta_1)$ is the poncaré duality of the suborbifold S^1 in $\mathbb{P}(3,3,3) \otimes S^1$, $e_A(E_{(\mathbf{g} \times 1)}) = e_A(E_{(\mathbf{g})})$, so from [1], we have:

$$
\int_{\mathbb{P}(3,3,3) \times S^1}^{orb} e_1^* \eta_1 \wedge e_A(E_{(\mathbf{g} \times 1)}) = \int_{\mathbb{P}(3,3,3)}^{orb} e_A(E_{(\mathbf{g})}) \qquad (4.3)
$$

The right side in (4.3) is calculated in the example of [7]. So the formula (4.2) can be calculated.

For the 3-multisector $Y_{(g_1^2 \times 1, g_1^2 \times 1, g_1^2 \times 1)} = \mathbb{P}(3,3,3) \times S^1$, let $\eta_i \in H^*(\mathbb{P}(3,3,3) \times S^1; \mathbb{Q})$ for $i = 1, 2, 3$. From (3.3), the dimension of the obstruction bundle $E_{(\mathbf{g} \times 1)}$ over $Y_{(g_1^2 \times 1, g_1^2 \times 1, g_1^2 \times 1)}$ is 2. The integration

$$
< \eta_1, \eta_2, \eta_3 >_{orb} := \int_{\mathbb{P}(3,3,3) \times S^1}^{orb} e_1^* \eta_1 \wedge e_2^* \eta_2 \wedge e_3^* \eta_3 \wedge e_A(E_{(\mathbf{g} \times 1)}) \qquad (4.4)
$$

is nonzero only if there are some $\eta_i \in H^1(\mathbb{P}(3,3,3) \times S^1; \mathbb{Q})$, $\eta_j \in H^2(\mathbb{P}(3,3,3) \times S^1; \mathbb{Q})$, because $H^1(\mathbb{P}(3,3,3) \times S^1; \mathbb{Q}) = H^0(\mathbb{P}(3,3,3); \mathbb{Q}) \otimes H^1(S^1; \mathbb{Q})$ and $H^2(\mathbb{P}(3,3,3) \times S^1; \mathbb{Q}) = H^2(\mathbb{P}(3,3,3); \mathbb{Q}) \otimes H^0(S^1; \mathbb{Q})$. Let $\eta_1 \in H^1(\mathbb{P}(3,3,3) \times S^1; \mathbb{Q}), \eta_2 \in H^2(\mathbb{P}(3,3,3) \times S^1; \mathbb{Q})$, $\eta_3 \in H^0(\mathbb{P}(3,3,3) \times S^1; \mathbb{Q})$. So the integration (4.4) is

$$
< \eta_1, \eta_2, \eta_3 >_{orb} := \int_{\mathbb{P}(3,3,3) \times S^1}^{orb} e_1^* \eta_1 \wedge e_2^* \eta_2 \wedge e_A(E_{(\mathbf{g} \times 1)}) \qquad (4.5)
$$

We know that $e_1^*\eta_1$ represents the class of $H^1(S^1;\mathbb{Q})$. $e_1^*(\eta_1)$ is the poncaré duality of the suborbifold S^1 in $\mathbb{P}(3,3,3)\otimes S^1$, $e_A(E_{(\mathbf{g}\times1)}) = e_A(E_{(\mathbf{g})})$, so from [1], we have:

$$\int_{\mathbb{P}(3,3,3)\times S^1}^{orb} e_1^*\eta_1 \wedge e_2^*\eta_2 \wedge e_A(E_{(\mathbf{g}\times1)}) = \int_{\mathbb{P}(3,3,3)}^{orb} e_2^*\eta_2 \wedge e_A(E_{(\mathbf{g})}) \qquad (4.6)$$

The right side in (4.6) is calculated in the example of [7]. So the formula (4.5) can be calculated. We complete the analysis of the Chen-Ruan ring structure of the orbifold $\mathbb{P}(1,2,2,3,3,3)\times S^1$.

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School of Mathematics, Peking University, Beijing 100871, P.R.China
E-mail address: dingfan@math.pku.edu.cn

Department of Mathematics, University of British Columbia, 1984 Mathematics Rd, Vancouver, BC, V6T 1Z2, Canada
E-mail address: jiangyf@math.ubc.ca

Institute of Mathematics, Academy of Mathematics and System Sciences, Chinese Academy of Sciences, Beijing 100080, P.R.China
E-mail address: pjz@mail.amss.ac.cn